

# Leader Selection via the Manipulability of Leader-Follower Networks

Hiroaki Kawashima and Magnus Egerstedt

**Abstract**—In this paper, we address the problem of selecting leaders in a network by investigating how much instantaneous impact the leaders have on the remaining agents. As a measurement of the influence of leaders' inputs, we exploit the notion of manipulability, which is recently developed for leader-follower networks driven by a state-dependent weighted consensus equation. This paper first extends the manipulability index in order to measure the influence of leaders' inputs on the network centroid. We then demonstrate in simulation how the manipulability index is suitable for selecting effective leaders.

## I. INTRODUCTION

Tracking of multiple agents to a given reference point while preserving interrelation among agents' states has been an important problem in robotics fields. For example, in applications including spacecrafts, unmanned aerial vehicles (UAVs), and indoor/outdoor mobile robots, it is often required for the agents to move toward a landmark or target point in formation.

The leader-follower approach has emerged in order to address this type of formation control problems [1]. In this approach, a single agent or multiple agents are selected as *leader(s)* that can inject control inputs to the network, while the remaining agents, which are referred to as *followers*, execute a simple protocol based on the states of adjacent agents. This approach provides a natural link between control theory and a networked agent with inputs. In particular, if the networked system is considered as a single system as a whole, and the followers run a consensus protocol, then the classical controllability notion in the linear system theory can be applied in a natural way [2], [3].

While the leader-follower approach has been widely used, a fundamental question still remains: how one should select the leaders out of the constituent agents, which is often referred to as the *leader-selection* problem. Once a leader or multiple leaders are selected, there may exist possible leaders' control to achieve a given task such as tracking or formation control. However, since the overall control performance is determined by the choice of the leaders, it is crucial to establish useful criteria for selecting the leaders.

Several indices have been recently introduced for leader-selection problems of networked systems, in particular under the linear consensus protocol. Some graph properties such as the degree of a vertex were studied in terms of the relation

to the leaders' performance [4]. The notion of network coherence, measured by the variance of the deviation from consensus, has been proposed in [5], and is also used for large-scale networks [6], [7]. A switching policy of leaders in order to improve the convergence rate of the velocities of the agents has been introduced for UAV control [8].

Under the nonlinear, state-dependent weighted consensus protocol, the previous indices are not directly applicable. An alternative yet straightforward approach could be to define a cost based on the predicted deviations of control points (e.g., centroid of agents) from given reference points. However, the network topology is often time-varying or state dependent. And, the surrounding environment may also change dynamically because of newly appeared obstacles or agents. Thus, leader-selection criteria using a predicted cost are not entirely appropriate for such dynamic situations.

Motivated by this reason, the present paper explores the use of an instantaneous measure, which is expected to be a useful criterion for selecting effective leaders under dynamic situations. In particular, we focus on using an index called *manipulability*, recently proposed in [9]. Similar to the original notion that measures the impact of the joint-angular velocities on the end-effectors in robot arms, the manipulability of leader-follower networks measures the instantaneous impact of leaders' inputs on the remaining agents. Since the index takes into account the network topologies, agent configurations, and input directions, simultaneously, it holds out promise to be a reasonable index for measuring the instantaneous effectiveness of leaders depending on a given situation. In this paper we demonstrate how the notion of manipulability can be applicable to the leader-selection problem when driving the centroid of agents to a given target state via the leaders' movements.

## II. LEADER-FOLLOWER NETWORKS

### A. Multi-agent Network with Leaders and Followers

Let  $x_i(t) \in \mathbb{R}^d$  ( $i = 1, \dots, N$ ) be the state of agent  $i$  at time  $t$ , the overall state (configuration) of the network is given by  $x(t) = [x_1^T(t), \dots, x_N^T(t)]^T \in \mathbb{R}^{Nd}$ . Consider that  $N_\ell$  out of  $N$  agents are assigned to be leaders, whose movements are considered as the inputs to the network. The remaining  $N_f (= N - N_\ell)$  agents are referred to as followers, each of which obeys a given control law.

We consider the situation where the interaction dynamics are defined through pairwise interactions. We say that when follower agents  $i$  and  $j$  are *connected*, then they share relative state information, and their pairwise control task is to maintain their distance  $\|x_i - x_j\|$  to a prespecified, positive value  $d_{ij}$ . If one of the agents in a connected pair is

H. Kawashima is with the Graduate School of Informatics, Kyoto University, Japan, a JSPS Postdoctoral Fellow for Research Abroad, and a visiting researcher at the School of Electrical and Computer Engineering, Georgia Institute of Technology kawashima@i.kyoto-u.ac.jp

M. Egerstedt is with the School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA 30308, USA magnus@gatech.edu

a leader agent and the other is a follower, then the follower's dynamics is designed so that it tries to maintain the distance.

Using a graph representation, the agents are described by nodes  $V = \{v_1, \dots, v_N\}$  and the connections between agents become edges  $E \subseteq V \times V$ , where the number of edges is  $M = |E|$  (the cardinality of  $E$ ). Then, the overall network is described by graph  $G = (V, E)$ . In this paper, we assume networks whose underlying graphs are undirected, static, and connected.<sup>1</sup>

### B. Notation for Leader and Follower Assignment

To explicitly denote the assignment of leaders and followers, we introduce the following notations. Let  $\ell : \{1, \dots, N_\ell\} \rightarrow \{1, \dots, N\}$  be an injective function whose image,  $\{\ell(i) | i = 1, \dots, N_\ell\}$ , is a set of leaders' indices. Let  $\delta_i$  be a vector whose  $i$ -th entry is 1 and all the remaining entries are 0s. Using the  $N \times N_\ell$  matrix  $\Delta_\ell \triangleq [\delta_{\ell(1)}, \dots, \delta_{\ell(N_\ell)}]$ , we can denote an indicator vector of leaders as  $\hat{\delta}_\ell \triangleq \Delta_\ell \mathbf{1}_{N_\ell}$  ( $= \sum_{i=1}^{N_\ell} \delta_{\ell(i)}$ ), where  $\mathbf{1}_p$  is a  $p$ -dimensional column vector with 1s in all its entries. Similarly, we define function  $f$ ,  $\Delta_f \triangleq [\delta_{f(1)}, \dots, \delta_{f(N_f)}]$ , and  $\hat{\delta}_f$  to indicate followers.  $P = [\Delta_f | \Delta_\ell]$  becomes a permutation matrix, which satisfies  $P^T P = P P^T = I_N$ , where  $I_p$  denotes the  $p \times p$  identity matrix. Besides, relations such as  $\Delta_\ell \Delta_\ell^T = \text{Diag}(\hat{\delta}_\ell)$ ,  $\Delta_\ell^T \Delta_\ell = I_{N_\ell}$ ,  $\Delta_\ell^T \mathbf{1}_N = \mathbf{1}_{N_\ell}$ , and  $\hat{\delta}_\ell^T \mathbf{1}_N = N_\ell$  will be used throughout the paper, where  $\text{Diag}(a)$  is the diagonal matrix whose diagonal is vector  $a$ .

Now, the states of leaders and followers can be grouped and denoted as vectors  $x_\ell(t) \in \mathbb{R}^{N_\ell d}$  and  $x_f(t) \in \mathbb{R}^{N_f d}$ , respectively:

$$\begin{aligned} x_\ell(t) &= [x_{\ell(1)}^T(t), \dots, x_{\ell(N_\ell)}^T(t)]^T = (\Delta_\ell^T \otimes I_d)x(t), \\ x_f(t) &= [x_{f(1)}^T(t), \dots, x_{f(N_f)}^T(t)]^T = (\Delta_f^T \otimes I_d)x(t), \end{aligned} \quad (1)$$

and

$$x(t) = (\Delta_\ell \otimes I_d)x_\ell(t) + (\Delta_f \otimes I_d)x_f(t), \quad (2)$$

where  $\otimes$  denotes the Kronecker product.

### C. Agent Dynamics

To formulate the followers' dynamics, we use a general, energy-based definition (e.g., [1]), which enables agents to achieve a distance-based formation control. Let

$$\mathcal{E}(x) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \mathcal{E}_{ij}(x_i(t), x_j(t)) \quad (3)$$

be the edge-tension energy, which is the summation of

$$\mathcal{E}_{ij}(x_i, x_j) = \begin{cases} \frac{1}{2} \{e_{ij}(\|x_i - x_j\|)\}^2 & (v_i, v_j) \in E \\ 0 & (v_i, v_j) \notin E, \end{cases} \quad (4)$$

where  $e_{ij} : \mathbb{R}^+ \rightarrow \mathbb{R}$  is a strictly increasing, twice differentiable function such that  $e_{ij}(d_{ij}) = 0$  ( $d_{ij} > 0$ ) and  $e'_{ij}(d_{ij}) \neq 0$ , where  $e'_{ij}(z) \triangleq \frac{de_{ij}(z)}{dz}$ .

<sup>1</sup>The assumption about static networks will only be used instantaneously. During the actual evolution of the system, the edge set will be allowed to vary over time.

Given the leaders' movements as the inputs to the network:

$$\dot{x}_\ell(t) = [\dot{x}_{\ell(1)}^T(t), \dots, \dot{x}_{\ell(N_\ell)}^T(t)]^T = u_\ell(t),$$

we define the dynamics of the followers such that each of the followers tries to minimize the related parts of the edge-tension energy (3) through a gradient descent direction:

$$\dot{x}_i(t) = - \sum_{j \in \mathcal{N}(i)} \frac{\partial \mathcal{E}_{ij}(x_i(t), x_j(t))}{\partial x_i} \quad (i = 1, \dots, N_f), \quad (5)$$

where  $\mathcal{N}(i) = \{j \in \{1, \dots, N\} \mid (v_i, v_j) \in E\}$  is the neighbor set of agent  $i$ . That is, the dynamics of the followers is designed such that each of the followers tries to maintain the desired distances to adjacent agents. Using the facts that  $\mathcal{E}_{ij} = \mathcal{E}_{ji}$  and  $\frac{\partial \mathcal{E}}{\partial x_i} = \frac{1}{2} \sum_{j \in \mathcal{N}(i)} \left( \frac{\partial \mathcal{E}_{ij}}{\partial x_i} + \frac{\partial \mathcal{E}_{ji}}{\partial x_i} \right)$ , the dynamics of all the followers can be denoted as

$$\dot{x}_f(t) = [\dot{x}_{f(1)}^T(t), \dots, \dot{x}_{f(N_f)}^T(t)]^T = - \frac{\partial \mathcal{E}(x)}{\partial x_f}. \quad (6)$$

Therefore, using this dynamics, the followers try to decrease (locally) the total energy (3) since  $\dot{\mathcal{E}} = \frac{\partial \mathcal{E}}{\partial x_f} \dot{x}_f + \frac{\partial \mathcal{E}}{\partial x_\ell} \dot{x}_\ell = -\|\frac{\partial \mathcal{E}}{\partial x_f}\|^2 + \frac{\partial \mathcal{E}}{\partial x_\ell} \dot{x}_\ell$ . In particular, if the leaders are not moving (i.e.,  $\dot{x}_\ell = 0$ ), the energy will not be increased by the followers, and will be decreased in many cases.

It can be easily shown that

$$\frac{\partial \mathcal{E}_{ij}(x_i, x_j)}{\partial x_i} = w_{ij}(\|x_i - x_j\|)(x_i - x_j)^T,$$

where  $w_{ij}(\|x_i - x_j\|) \triangleq \{e_{ij}(\|x_i - x_j\|)e'_{ij}(\|x_i - x_j\|)\} / \|x_i - x_j\|$ . Thus, (5) becomes a state-dependent weighted consensus equation [1]. Let  $D \in \mathbb{R}^{N \times M}$  be the incidence matrix of graph  $G$  with an arbitrary but consistent assignment of the orientation on the edges. Let  $W(x) \in \mathbb{R}^{M \times M}$  be the diagonal weight matrix whose  $k$ -th element is  $[W(x)]_{kk} = w_{i_k j_k}(\|x_{i_k} - x_{j_k}\|)$ , where  $i_k$  and  $j_k$  are the agents connected by edge  $k$ . Then, the weighted graph Laplacian of  $G$  becomes  $L_w(x) = DW(x)D^T \in \mathbb{R}^{N \times N}$ . Here, the dynamics with assigning all the agents to followers becomes  $\dot{x} = -(L_w \otimes I_d)x$ . Therefore, noting the relation  $(X \otimes I_d)(Y \otimes I_d) = XY \otimes I_d$  and using (1), we can rewrite (6) as

$$\dot{x}_f(t) = -((\Delta_f^T L_w) \otimes I_d)x(t). \quad (7)$$

Eventually, the dynamics of overall agents becomes

$$\dot{x} = -((\text{Diag}(\hat{\delta}_f) L_w) \otimes I_d)x + (\Delta_\ell \otimes I_d)u_\ell, \quad (8)$$

which can be denoted as  $\dot{x} = F_\ell(x, u_\ell)$ . In case the assignment of leaders changes dynamically,  $\ell$  becomes a time-varying function, and the networked system is considered as a switched system as a whole.

## III. LEADER SELECTION FOR TRACKING

### A. Closed-Loop Tracking

Consider the task of driving the centroid of the agents,  $\bar{x}(t) = \frac{1}{N} \sum_{i=1}^N x_i(t) = \frac{1}{N} (\mathbf{1}_N^T \otimes I_d)x(t)$ , to a given

reference point,  $x_r \in \mathbb{R}^d$ , with the agent dynamics (8). We here use the input

$$u_\ell(t) = (\mathbf{1}_{N_\ell} \otimes I_d) \tilde{u}_\ell(t), \quad (9)$$

to achieve the proportional regulation of the centroid with gain  $k > 0$ , where every leader has the same input

$$\tilde{u}_\ell(t) \triangleq \frac{N}{N_\ell} k(x_r - \bar{x}) - \frac{1}{N_\ell} ((\hat{\delta}_\ell^T L_w) \otimes I_d) x. \quad (10)$$

Since it is natural to assume in many applications that each agent has the limit on its input norm, we constrain each leader's velocity as  $\|\tilde{u}_\ell(t)\| = v_c$ , where  $v_c$  is a given constant. Solving this equation, we get  $k$  as a function of  $x$  and  $\hat{\delta}_\ell$ . Therefore, in what follows we denote  $k$  as  $k_\ell(x)$ . Assume that  $k_\ell(x)$  is obtained as a positive real value. Then, the dynamics (8) with the closed-loop feedback (9) results in the autonomous system:

$$\dot{x} = \hat{F}_\ell(x) \triangleq - \left[ \left( \text{Diag}(\hat{\delta}_f) + \frac{1}{N_\ell} \hat{\delta}_\ell \hat{\delta}_\ell^T \right) L_w(x) \right] \otimes I_d x + \frac{N}{N_\ell} (\hat{\delta}_\ell \otimes I_d) k_\ell(x) (x_r - \bar{x}). \quad (11)$$

Multiplying  $\frac{1}{N} (\mathbf{1}_N^T \otimes I_d)$  from left and noting  $\mathbf{1}_N^T L_w = 0$ , we have  $\dot{\bar{x}} = k_\ell(x) (x_r - \bar{x})$ . In this paper we use this particular dynamics, while other inputs can also be used to perform the similar tracking. Since the permutation of the indices in the image of  $\ell$  does not affect  $\hat{F}_\ell(x)$ , in what follows we denote the assignment of leaders with an indicator vector  $\hat{\delta}_\ell$ .

We now remark several facts regarding the assumption on the gain  $k_\ell(x)$ . Letting  $\alpha(x) \triangleq -\frac{1}{N_\ell} ((\hat{\delta}_\ell^T L_w) \otimes I_d) x$  and  $\beta(x) \triangleq \frac{N}{N_\ell} (x_r - \bar{x})$ , we rewrite (10) as  $\tilde{u}_\ell = \alpha(x) + k\beta(x)$ . Thus,  $k$  is given by one of the solutions of  $\|\beta\|^2 k^2 + 2(\alpha^T \beta)k + \|\alpha\|^2 - v_c^2 = 0$ , and it is natural to choose the larger solution as  $k_\ell(x)$ . If  $\|\alpha\| < v_c$ ,  $k_\ell(x)$  is always positive; otherwise, the condition for positive gain is given by  $\|\alpha\| \sin \theta \leq v_c \leq \|\alpha\|$  with  $\alpha^T \beta < 0$ , where  $\theta$  is the angle between  $\alpha$  and  $\beta$ ; that is,  $\theta$  needs to be close to  $\pi$ .

Recall that the term  $\alpha$  is derived from the weighted consensus; and it thus tries to compensate the increased energy  $\mathcal{E}$ . In many situations, there exist some leader assignments that satisfy the latter condition even when  $\|\alpha\| \geq v_c$ . This is because  $\alpha$  can take variety of directions depending on the assignments of leaders, and some of them may have roughly the opposite directions to  $\beta$ . However, if none of the leader sets yields an admissible  $k$ , one can run the original consensus protocol until  $\|\alpha\|$  becomes small enough to satisfy  $k > 0$ . The following remark holds in the extreme situation when  $\alpha = 0$ .

*Remark 3.1:* If all the desired distances are satisfied, i.e.,  $\|x_i - x_j\| = d_{ij} \quad \forall (v_i, v_j) \in \mathbf{E}$ , then  $k_\ell(x)$  does not depend on assignment  $\hat{\delta}_\ell$  under given  $N_\ell$ , since  $k_\ell(x) = \frac{N_\ell v_c}{N \|x_r - \bar{x}\|}$ .

### B. Leader-Selection Problem

Suppose that the autonomous system (11) is used for the tracking task, we formulate the leader-selection via the prediction cost

$$\mathcal{J}(\hat{\delta}_\ell) = \int_t^{t+T} \|x_r - \bar{x}(s)\|^2 ds, \quad (12)$$

as the problem of finding the best leader assignment

$$\hat{\delta}_\ell^* = \arg \min_{\hat{\delta}_\ell \in \mathcal{L}} \mathcal{J}(\hat{\delta}_\ell) \quad \text{s.t.} \quad \|\hat{\delta}_\ell\|_1 = N_\ell, \quad (13)$$

where  $\mathcal{L} = \{\hat{\delta}_\ell \mid k_\ell(x) \text{ is positive real}\}$ , and  $\|\hat{\delta}_\ell\|_1 (= \|\hat{\delta}_\ell\|_0)$  is the number of leaders.

Here, the cost  $\mathcal{J}$  is defined based on the assumption that the environment and the network topology will not be changed during the horizon  $[t, t+T]$ . Therefore, a reliable prediction becomes difficult under dynamic situations. For this reason, we take a second approach to select leaders. That is, we introduce a measurement to evaluate the instantaneous impact of the leaders' inputs on the network, which will be explained in the next section.

## IV. LEADER SELECTION VIA MANIPULABILITY

### A. Manipulability of Leader-Follower Networks

The manipulability of leader-follower networks [9] is a measure to evaluate the influence of the leaders' movements on the remaining of the network. Similar to the original notion used in the field of robot-arm manipulators [10], [11], the manipulability index of leader-follower networks is defined as the ratio between the norm of response (followers' motion, in our case) to that of inputs (leaders' motion):

$$R(x, \mathbf{E}, \dot{x}_\ell) = \frac{\dot{x}_f^T Q_f \dot{x}_f}{\dot{x}_\ell^T Q_\ell \dot{x}_\ell}, \quad (14)$$

where  $Q_f = Q_f^T \succ 0$  and  $Q_\ell = Q_\ell^T \succ 0$  are positive definite weight matrices. In what follows, we use  $Q_f = I_{N_f d}$  and  $Q_\ell = I_{N_\ell d}$  for simplicity.

We here extend the index in order to measure the response of the centroid to the leaders' movements.

*Definition 4.1:* Given a leader-follower network whose configuration is  $x$  and the underlying graph is  $G = (\mathbf{V}, \mathbf{E})$ .

$$R_e(x, \mathbf{E}, \dot{x}_\ell) = \frac{\|\dot{\bar{x}}\|^2}{\|\dot{x}_\ell\|^2} \quad (15)$$

is the *ensemble manipulability* of the network under the leaders' motion  $\dot{x}_\ell$ .

We formulate the leader selection via the ensemble manipulability as

$$\hat{\delta}_\ell^* = \arg \max_{\hat{\delta}_\ell \in \mathcal{L}} R_e \quad \text{s.t.} \quad \|\hat{\delta}_\ell\|_1 = N_\ell, \quad (16)$$

where similar dynamics and inputs as (13) are assumed.

Noting that  $\frac{d\|x_r - \bar{x}\|^2}{dt} = -2(x_r - \bar{x})^T \dot{\bar{x}} = -2k_\ell(x) \|x_r - \bar{x}\|^2 < 0$ , we know that  $\|x_r - \bar{x}\|^2$  is monotonously decreasing under the input (9). Then, the assignment  $\hat{\delta}_\ell$  that achieves  $\min \frac{d\|x_r - \bar{x}\|^2}{dt}$  is a reasonable choice to the extent of instantaneous decision. Here the following relation holds:

$$\arg \max_{\hat{\delta}_\ell \in \mathcal{L}} R_e = \arg \max_{\hat{\delta}_\ell \in \mathcal{L}} k_\ell(x) = \arg \min_{\hat{\delta}_\ell \in \mathcal{L}} \frac{d\|x_r - \bar{x}\|^2}{dt}.$$

This implies that we can use the notion of ensemble manipulability for the leader-selection problem since it finds the best leader assignment in terms of achieving the steepest descent of  $\|x_r - \bar{x}\|^2$ .

Some difficulty arises here. As discussed in [9], since  $\dot{x}_f = -\frac{\partial \mathcal{E}^T}{\partial x_f}$  is a function of  $x_f$  and  $x_\ell$  but not  $\dot{x}_\ell$ , we need to integrate over time to see the influence of  $\dot{x}_\ell$ . However, the leaders' motion  $\dot{x}_\ell$  can change on the time interval of the integration. Moreover, if the desired distances are perfectly realized, i.e.,  $\|x_i - x_j\| = d_{ij} \quad \forall (v_i, v_j) \in \mathbb{E}$ , it then follows from Remark 3.1 that  $R_e$  takes the same value  $R_e = \frac{k_\ell(x)^2 \|x_f - \bar{x}\|^2}{N_\ell v^2} = \frac{N_\ell}{N^2}$  for any leader assignment. This implies that  $R_e$  does not provide any information for selecting leaders when  $\mathcal{E} = 0$  is satisfied. In order to evaluate the instantaneous influence of the leaders' input  $\dot{x}_\ell$  without using any integral action, we introduce the approximation of the followers' dynamics and the approximate notion of manipulability proposed in [9].

### B. Approximate Dynamics and Manipulability

**Definition 4.2:** [9, Definition 4.1] The *rigid-link approximation* of the dynamics in a given leader-follower network is the ideal situation when all the given desired distances  $\{d_{ij}\}_{(v_i, v_j) \in \mathbb{E}}$  are perfectly maintained by the followers (i.e.,  $\|x_i - x_j\| = d_{ij} \quad \forall (v_i, v_j) \in \mathbb{E}$ ).

This approximation is reasonable unless leaders move much faster than followers. Under this approximation, the following has been proven in [9]. Let  $R(x) \in \mathbb{R}^{M \times Nd}$  be the rigidity matrix of the given state  $x$  and the underlying graph  $G$  [12], [13]. Here,  $R$  consists of  $M \times N$  blocks of  $1 \times d$  row vectors, where its  $(k, i_k)$  and  $(k, j_k)$  blocks are  $(x_{i_k} - x_{j_k})^T$  and  $-(x_{i_k} - x_{j_k})^T$  (or  $-(x_{i_k} - x_{j_k})^T$  and  $(x_{i_k} - x_{j_k})^T$ ), respectively; where,  $i_k$  and  $j_k$  are the agents connected by edge  $k$ . Let us define matrices  $R_f(x) \in \mathbb{R}^{M \times N_f d}$  and  $R_\ell(x) \in \mathbb{R}^{M \times N_\ell d}$  as

$$R_\ell(x) \triangleq R(x)(\Delta_\ell \otimes I_d), \quad R_f(x) \triangleq R(x)(\Delta_f \otimes I_d). \quad (17)$$

**Example 4.1:** In the case the last indices of  $\{1, \dots, N\}$  are assigned to the leaders, i.e.,  $\ell(i) = N_f + i \quad (i = 1, \dots, N_\ell)$  and  $f(i) = i \quad (i = 1, \dots, N_f)$ ,  $R_\ell$  and  $R_f$  are given by the submatrices of  $R$  such that  $R = [R_f | R_\ell]$ .

Using these notations, the dynamics of the followers under the rigid-link approximation becomes

$$\dot{x}_f = J\dot{x}_\ell = -R_f^\dagger R_\ell \dot{x}_\ell, \quad (18)$$

where  $J(x) \triangleq -R_f^\dagger R_\ell$ , and  $R_f^\dagger$  is the Moore-Penrose pseudo inverse of  $R_f$ . We also assumed that the motions of the leaders,  $\dot{x}_{\ell(1)}, \dots, \dot{x}_{\ell(N_\ell)}$ , are properly constrained so as not to breakdown the rigid-link approximation.<sup>2</sup> Note that  $R, R_f, R_\ell$ , and  $J$  also depend on the network topology  $\mathbb{E}$ .

Substituting (18) into (14) yields

$$\hat{m}(x, \mathbb{E}, \dot{x}_\ell) = \frac{\dot{x}_\ell^T J^T J \dot{x}_\ell}{\dot{x}_\ell^T \dot{x}_\ell}. \quad (19)$$

This index is referred to as the *approximate manipulability* in [9]; in particular, the identity matrices are used here for  $Q_f$

<sup>2</sup>This condition is shown and referred to as a *feasible leader motion* in [9], which can be derived from the notion of *feasible motion* given in [14]. For example,  $\dot{x}_\ell = (\mathbf{1}_{N_\ell} \otimes I_d) \dot{\hat{x}}_\ell$  is always a feasible leader motion.

and  $Q_\ell$ . The approximate manipulability provides a short-term estimate of the influence of the leaders' motion  $\dot{x}_\ell$  on the followers' motion. Here, its maximum value with respect to  $\dot{x}_\ell$  can be obtained as the maximum eigenvalue of  $J^T J$  since (19) has a form of the Rayleigh quotient.

Similar to the approximation of the manipulability, the *approximate ensemble manipulability* can be derived by substituting (18) into (15). Here, we assume that all the leaders take the same motion, i.e.,  $\dot{x}_\ell = (\mathbf{1}_{N_\ell} \otimes I_d) \dot{\hat{x}}_\ell$ , since we will use the particular leaders' input given in (9). Moreover, as shown in the following, this assumption also leads to a form of the Rayleigh quotient.

**Proposition 4.1:** Given the leaders' motion  $\dot{x}_\ell = (\mathbf{1}_{N_\ell} \otimes I_d) \dot{\hat{x}}_\ell$ , the approximate ensemble manipulability under the rigid-link approximation (Definition 4.2) is given by

$$\hat{m}_e(x, \mathbb{E}, \dot{x}_\ell) = \frac{\dot{\hat{x}}_\ell^T (\tilde{J}^T \tilde{J} + N_\ell I_d)^2 \dot{\hat{x}}_\ell}{N^2 N_\ell \dot{\hat{x}}_\ell^T \dot{\hat{x}}_\ell}, \quad (20)$$

where  $\tilde{J} \triangleq -R_f^\dagger R_\ell (\mathbf{1}_{N_\ell} \otimes I_d) = R_f^\dagger R_f (\mathbf{1}_{N_f} \otimes I_d)$ .

**Proof:** Using (2), (18), and  $\bar{x} = \frac{1}{N} (\mathbf{1}_N^T \otimes I_d) x$ , we get

$$\begin{aligned} \dot{\hat{x}} &= \frac{1}{N} ((\mathbf{1}_{N_f}^T \otimes I_d) \dot{x}_f + (\mathbf{1}_{N_\ell}^T \otimes I_d) \dot{x}_\ell) \\ &= \frac{1}{N} (-(\mathbf{1}_{N_f}^T \otimes I_d) R_f^\dagger R_\ell \dot{x}_\ell + (\mathbf{1}_{N_\ell}^T \otimes I_d) \dot{x}_\ell) \\ &= \frac{1}{N} ((\mathbf{1}_{N_f}^T \otimes I_d) R_f^\dagger R_f (\mathbf{1}_{N_f} \otimes I_d) + N_\ell I_d) \dot{\hat{x}}_\ell. \end{aligned}$$

The last equality follows from the assumption  $\dot{x}_\ell = (\mathbf{1}_{N_\ell} \otimes I_d) \dot{\hat{x}}_\ell$  and the fact that  $R_\ell (\mathbf{1}_{N_\ell} \otimes I_d) = -R_f (\mathbf{1}_{N_f} \otimes I_d)$ , which can be shown from the definition of the rigidity matrix. Using the fact that  $(R_f^\dagger R_f)^2 = R_f^\dagger R_f$ , we obtain  $(\mathbf{1}_{N_f}^T \otimes I_d) R_f^\dagger R_f (\mathbf{1}_{N_f} \otimes I_d) = \tilde{J}^T \tilde{J}$ , and (20) follows. ■

**Example 4.2:** If  $N_\ell = 1$ , then  $\tilde{J} = J$ ,  $\dot{\hat{x}}_\ell = \dot{x}_\ell$ , and

$$\hat{m}_e(x, \mathbb{E}, \dot{x}_\ell) = \frac{\dot{x}_\ell^T (J^T J + I_d)^2 \dot{x}_\ell}{N^2 \dot{x}_\ell^T \dot{x}_\ell}. \quad (21)$$

Using (20), we can derive the following property.

**Proposition 4.2:** Under the same assumption as in Proposition 4.1, the approximate ensemble manipulability  $\hat{m}_e$  takes

$$0 < \hat{m}_e \leq \frac{1}{N_\ell}. \quad (22)$$

**Proof:** Let  $P \triangleq \frac{1}{N^2 N_\ell} (\tilde{J}^T \tilde{J} + N_\ell I_d)^2$ .  $\hat{m}_e \geq \min_{\dot{\hat{x}}_\ell} \hat{m}_e = \lambda_{\min}(P) > 0$  follows from the facts that (20) has the form of the Rayleigh quotient with  $P$  and that  $P$  is positive definite. Similarly,  $\hat{m}_e \leq \max_{\dot{\hat{x}}_\ell} \hat{m}_e = \lambda_{\max}(P)$ . Meanwhile,  $\lambda_{\max}(\tilde{J}^T \tilde{J} + N_\ell I_d) = \lambda_{\max}((\mathbf{1}_{N_f}^T \otimes I_d) R_f^\dagger R_f (\mathbf{1}_{N_f} \otimes I_d) + N_\ell I_d) \leq \lambda_{\max}((\mathbf{1}_{N_f}^T \otimes I_d) (\mathbf{1}_{N_f} \otimes I_d) + N_\ell I_d) = \lambda_{\max}(N_f I_d + N_\ell I_d) = N$ . Thus,  $\lambda_{\max}((\tilde{J}^T \tilde{J} + N_\ell I_d)^2) \leq N^2$ , and  $\lambda_{\max}(P) \leq 1/N_\ell$  follows. ■

We will use this approximate ensemble manipulability  $\hat{m}_e$  instead of  $m_e$  for the leader selection formulated in (16).

## V. EXAMPLES

This section demonstrates how the approximate ensemble manipulability finds a reasonable leader assignment in terms

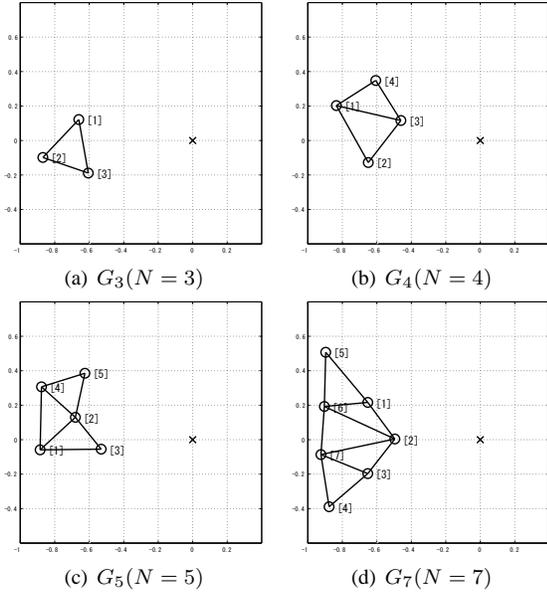


Fig. 1. Networks used in the example. The circles depict the agents' states and the lines depict the connections between agents. The numbers in the brackets are agent indices. The reference point is  $x_r = [0, 0]^T$ , depicted by a cross in each figure.

of driving the centroid of the agents to a given reference point. In the following examples, we focus on the case of  $N_\ell = 1$  for the sake of illustrating the basic characteristics of the proposed index. We first compare the manipulability index with the cost defined in (12) in terms of how consistent the selected leaders are. Then, we show examples of the tracking task when the leader is allowed to be switched dynamically.

In the simulation,  $d = 2$  was used for the dimensionality of the state space. Thus, in this example, the states of agents are depicted as points on the 2- $d$  plane. The followers' dynamics were given by (6) with  $e_{ij}(\|x_i - x_j\|) = c(\|x_i - x_j\| - d_{ij})$ , where  $c = 5\sqrt{2}$  was used to ensure that the rigid-link approximation is almost valid. Meanwhile, the leaders' dynamics was given by (9) with constraint  $v_c = 1$ . The reference point was set to  $x_r = [0, 0]^T$  for all the examples.

#### A. Comparison of Predicted Cost $\mathcal{J}$ and Manipulability $\hat{m}_e$

Four networks with different underlying graphs  $G_3$ ,  $G_4$ ,  $G_5$ , and  $G_7$  were prepared. Fig. 1 shows their configurations (i.e., state  $x$ ) and network topologies (i.e., edge set  $E$ ). Throughout this example, we use the name of the underlying graph to refer each network. The desired distances,  $d_{ij} \forall (v_i, v_j) \in E$ , were satisfied in the configurations depicted in the figures. In each of these networks, we calculated the cost  $\mathcal{J}(i)$  given in (12) and the approximate ensemble manipulability  $\hat{m}_e(i)$  for each agent  $i \in \{1, \dots, N\}$ , where we simply use the index of the agent in the arguments of  $\mathcal{J}$  and  $\hat{m}_e$ , since we assumed  $N_\ell = 1$ . A short enough time horizon,  $T = 0.2$ , was chosen to calculate  $\mathcal{J}(i)$  as we focus on evaluating short-term effects of leaders' inputs.

Table I shows the comparison between  $\mathcal{J}(i)$  and  $\hat{m}_e(i)$  in each of the networks. In each table, the values in the

TABLE I  
COMPARISON BETWEEN  $\mathcal{J}(i)$  AND  $\hat{m}_e(i)$

Network $G_3$				
Agent	$\mathcal{J}(i) [\times 10^{-2}]$	(ascending)	$\hat{m}_e(i)$	(descending)
1	8.858	(3)	0.315	(3)
2	8.186	(1)	0.974	(1)
3	8.650	(2)	0.489	(2)
Network $G_4$				
Agent	$\mathcal{J}(i) [\times 10^{-2}]$	(ascending)	$\hat{m}_e(i)$	(descending)
1	6.751	(2)	0.989	(2)
2	7.538	(4)	0.189	(4)
3	6.734	(1)	0.993	(1)
4	7.379	(3)	0.259	(3)
Network $G_5$				
Agent	$\mathcal{J}(i) [\times 10^{-2}]$	(ascending)	$\hat{m}_e(i)$	(descending)
1	9.618	(4)	0.375	(4)
2	9.115	(1)	0.999	(1)
3	9.261	(2)	0.727	(3)
4	9.318	(3)	0.751	(2)
5	9.755	(5)	0.214	(5)
Network $G_7$				
Agent	$\mathcal{J}(i) [\times 10^{-2}]$	(ascending)	$\hat{m}_e(i)$	(descending)
1	5.740	(5)	0.604	(4)
2	5.628	(1)	0.997	(1)
3	5.728	(4)	0.514	(5)
4	5.815	(6)	0.152	(7)
5	5.832	(7)	0.170	(6)
6	5.712	(3)	0.728	(3)
7	5.705	(2)	0.793	(2)

parentheses denote the rank of each value in the ascending order of  $\mathcal{J}(i)$  or in the descending order of  $\hat{m}_e(i)$ . Therefore, the agent that has the first rank will be selected as the leader. We see that, in each network, the same leader is selected with both criteria. In addition, not only the first rank but the ordering of the values is almost consistent between  $\mathcal{J}$  and  $\hat{m}_e$ . While the orders were switched between rank 2 and 3 in  $G_5$ , between rank 4 and 5 in  $G_7$ , and between rank 6 and 7 in  $G_7$ , the values corresponding to these pairs of ranks are relatively close each other in the both criteria. Hence, these examples indicate that the approximate ensemble manipulability  $\hat{m}_e$  can be an alternative of the predicted cost  $\mathcal{J}$  that involves an integral action.

Recall that the original ensemble manipulability  $m_e$  cannot be used in these configurations to compare the agents; that is,  $m_e$  takes the same value for every agent, since all the desired distances are satisfied here (Remark 3.1). Therefore, this result illustrates the advantage of the approximation introduced in Section IV.

#### B. Online Leader Selection

In this experiment, the approximate ensemble manipulability  $\hat{m}_e(i)$  was calculated and compared continually in order to perform the online selection of leaders. Specifically, the agent that gave the maximum  $\hat{m}_e$  at each time point was selected as the leader. The desired distances were set based on the initial configuration.

Fig. 2 (a) shows an example, in which no leader switch occurred. Here, the selected leader is depicted by a filled circle. Fig. 2 (b) shows the temporal change of  $\hat{m}_e(i)$  ( $i =$

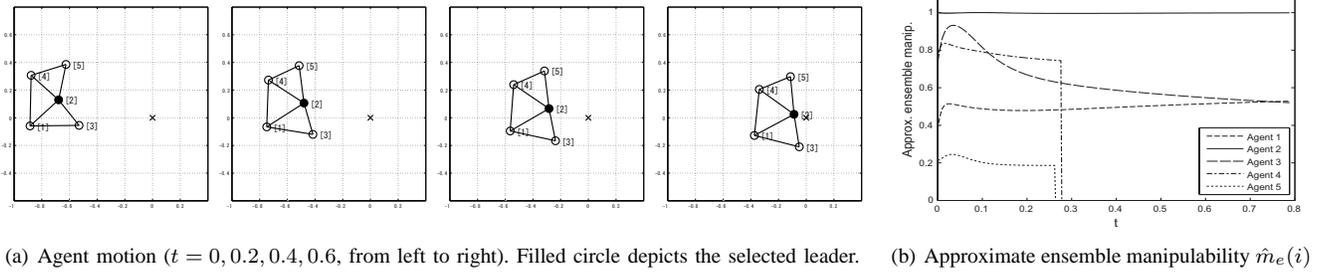


Fig. 2. Example of agent motion with dynamic leader selection. Agent 2 always stayed as the leader and no leader switching occurred.

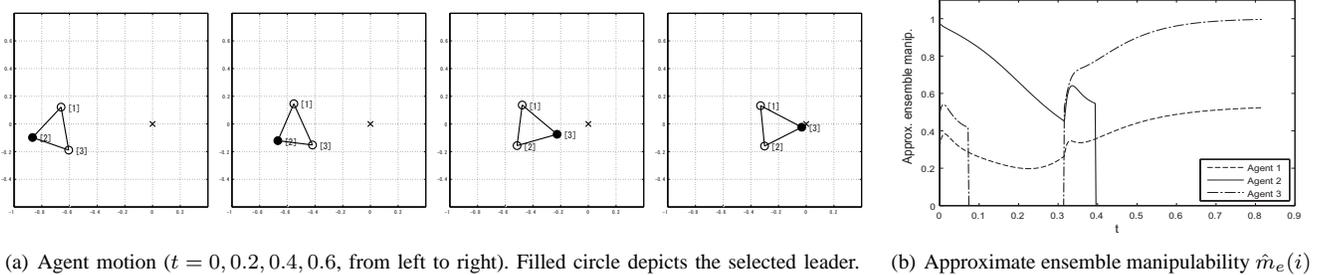


Fig. 3. Example of agent motion with dynamic leader selection. The switching of leaders from agent 2 to agent 3 occurred around  $t = 0.32$ .

1, ..., 5) during the agents' motion shown in Fig. 2 (a). The final time was chosen as the time when  $\|x_r - x(t)\|^2 = \epsilon$  was achieved, where  $\epsilon = 0.01$  was used. We see that all the values take between 0 and 1, which is in accord with Proposition 4.2, and in particular the agent 2, selected as the leader, takes almost  $\hat{R}_e = 1$  all the time in this example. When agent  $i$  was not able to satisfy the constraints  $\|\dot{x}_\ell\| = v_c$ , i.e., does not yields a positive real gain  $k_i(x)$ , we set the manipulability of agent  $i$  to  $\hat{m}_e(i) = 0$ . From the figure, we see that two agents (agent 4 and 5) could not achieve this constraints from around  $t = 0.27$ .

Fig. 3 (a) shows another example, in which the leaders was once switched from agent 2 to agent 3. From the value  $\hat{m}_e(i)$  ( $i = 1, 2, 3$ ) shown in Fig. 3 (b), we observe that  $\hat{m}_e(2)$  decreased in the first part and that finally the leader was switched to agent 3 around  $t = 0.32$ . This example shows the characteristics of the proposed index that it can take into account the difference of agent configurations and adaptively change the leader assignment depending on the situations.

## VI. CONCLUSION

This paper addressed the problem of selecting leaders in leader-follower networks by using the notion of manipulability, an index to estimate how injected leaders' inputs influence the network in short-term. We introduced the ensemble manipulability as an extended index to measure the influence on the centroid of the agents, and we demonstrated the manipulability-based leader selection for driving the centroid of agents to a given reference point in simulation.

## REFERENCES

- [1] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.
- [2] H. G. Tanner, "On the controllability of nearest neighbor interconnections," *IEEE Conference on Decision and Control*, pp. 2467–2472, 2004.
- [3] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt., "Controllability of multi-agent systems from a graph-theoretic perspective," *SIAM Journal on Control and Optimization*, vol. 48, no. 1, pp. 162–186., 2009.
- [4] T. Borsche and S. Attia, "On leader election in multi-agent control systems," *IEEE Conference on Chinese Control and Decision Conference*, pp. 102–107, 2010.
- [5] S. Patterson and B. Bamieh, "Leader selection for optimal network coherence," *IEEE Conference on Decision and Control*, pp. 2692–2697, 2010.
- [6] M. Fardad, F. Lin, and M. Jovanović, "Algorithms for leader selection in large dynamical networks: Noise-free leaders," *IEEE Conference on Decision and Control and European Control Conference*, pp. 7188–7193, 2011.
- [7] F. Lin, M. Fardad, and M. Jovanović, "Algorithms for leader selection in large dynamical networks: Noise-corrupted leaders," *IEEE Conference on Decision and Control and European Control Conference*, pp. 2932–2937, 2011.
- [8] A. Franchi, H. Bühlhoff, and P. Giordano, "Distributed online leader selection in the bilateral teleoperation of multiple UAVs," *IEEE Conference on Decision and Control and European Control Conference*, pp. 3559–3565, 2011.
- [9] H. Kawashima and M. Egerstedt, "Approximate manipulability of leader-follower networks," *IEEE Conference on Decision and Control and European Control Conference*, pp. 6618–6623, 2011.
- [10] A. Bicchi, C. Melchiorri, and D. Balluchi, "On the mobility and manipulability of general multiple limb robots," *IEEE Trans. on Robotics and Automation*, vol. 11, no. 2, pp. 215–228, 1995.
- [11] A. Bicchi and D. Prattichizzo, "Manipulability of cooperating robots with unactuated joints and closed-chain mechanisms," *IEEE Trans. on Robotics and Automation*, vol. 16, no. 4, pp. 336–345, 2000.
- [12] B. Roth, "Rigid and flexible frameworks," *The American Mathematical Monthly*, vol. 88, no. 1, pp. 6–21, 1981.
- [13] T. Eren and P. Belhumeur, "A framework for maintaining formations based on rigidity," *Proc. 15th IFAC World Congress*, pp. 2752–2757, 2002.
- [14] P. Tabuada, G. Pappas, and P. Lima, "Motion feasibility of multi-agent formations," *IEEE Trans. on Robotics*, vol. 21, no. 3, pp. 387–392, 2005.